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TORUS ACTIONS ON A PRODUCT OF TWO ODD SPHERES

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§1. INTRODUCTION

THE OBJECT of this paper is to describe certain methods which may be applied to the study of arbitrary topological torus actions, and to apply these methods to actions on a product of two odd spheres. We are motivated by certain recent papers of Wu-Yi Hsiang ([4], [5] and others), which study actions of a compact connected Lie group on Euclidean spaces, spheres, and complex projective spaces by restricting an action to a maximal torus of the group and applying certain known results concerning torus actions on these spaces. It is hoped that the methods and results of this paper may be useful in obtaining results about actions of compact connected Lie groups on other spaces.

The most important of the known results used by Hsiang is the following theorem of Borel ([1], p. 175). (See Section two for notation.)

THEOREM. *If a torus T acts on $X \sim S^n$, then $n - \dim F(T) = \Sigma_H[\dim F(H) - \dim F(T)]$ where H runs through the corank one subtori of T.*

This paper will give certain generalizations of Borel's theorem. Let us make the following definition.

DEFINITION 1.1. *For X a topological space, let $c(X)$ be the smallest integer such that any torus acting on X has some subtorus of corank $c(X)$ or less that has fixed points.*

Borel's theorem shows that if $X \sim S^n$, then $c(X) \leq 1$, and further that there is a formula relating the fixed point sets of the various subtori of T of corank zero or one. These are the kinds of results which we will generalize. Our methods will enable us to find $c(X)$ and a formula involving the fixed point sets of the various subtori of T of corank $0, 1, \ldots, c(X)$ for spaces X whose rational cohomology rings are sufficiently simple. To state our major result, we will need the following notation: for $X \sim S^p \times S^q$ with p and q odd, let $e(X) = (p+1)(q+1)/4$. Let $e(\varnothing) = 0$. It is easy to show that if a torus T acts on $X \sim S^p \times S^q$ with p and q odd, then either the fixed point set $F(T)$ is empty, or else $F(T) \sim S^k \times S^l$ with k and l odd. (We must have $\chi(F(T)) = \chi(X) = 0$, $\dim H^*(F(T)) = \dim H^*(X) = 4$, and, by [2], Theorem 6.1, the cohomology of any component of $F(T)$ must satisfy Poincaré duality and have its highest dimensional elements in an even dimension.)

THEOREM 1.2. *Suppose that a torus T acts topologically on $X \sim S^p \times S^q$, p and q odd. Suppose that $F(T) = \varnothing$. Then*

$$e(X) - eF(T) - \Sigma_H[eF(H) - eF(T)] = \Sigma_K\{eF(K) - eF(T) - \Sigma_{H \supseteq K}[eF(H) - eF(T)]\}$$

where H runs through the subtori of corank one and K runs through the subtori of corank two in T.

CONJECTURE 1.3. *The theorem above holds without the hypothesis that $F(T) = \emptyset$.*

Note that the terms $eF(T)$ are all zero in 1.2. They are included only to make the statement of 1.3 easy.

We can use theorem 1.2 to prove some simple facts about torus actions on a product of two odd spheres. Suppose we have an action of a torus T on a space X. We will say that a subtorus H of T is *distinguished* if $F(H) \supsetneq F(L)$ for any subtorus L of T such that $H \subsetneq L$. (In particular, if $H \neq T$ then $F(H) \neq \emptyset$.) In the course of proving 1.2, we will see that only distinguished H's and K's will appear nontrivially in the summations in 1.2 or 1.3.

PROPOSITION 1.4. *If a torus T acts topologically on X orientable with $X \sim S^p \times S^q$ with p and q odd, then the identity component of the ineffective kernal is equal to the identity component of the intersection of all the distinguished subtori of corank one and two. Further, the identity component of any isotropy subgroup is equal to the identity component of some intersection of distinguished subtori of corank one and two.*

In Section two below, we develope a method for finding $c(X)$ and the parallels of Borel's theorem. In Section three, we prove 1.2 and 1.4. In Section four we briefly describe the reduction of 1.3 to an algebraic problem.

One should note that although the present paper is entirely limited to the study of torus actions, the methods will be useful for the study of actions of groups of the type $(Z_p)^k$.

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§2. GENERAL RESULTS

Transformation groups will be studied here in the "Borel setting"; that is, the chief tool will be the diagram $X' \xleftarrow{\rho} X_G \xrightarrow{\pi} B_G$ and its cohomology. Here, G acts on X, X' is the orbit space X/G, $B_G = B(G)$ the universal classifying space of G, and X_G is the balanced product $X \times_G E_G$, where E_G is the universal total space of G. In this paper G will always be a torus T. Cohomology will always be Alexander–Spanier cohomology with compact supports and coefficients in the rationals $\mathbf{Q}$. (The compact supports and rational coefficients will not usually be indicated.) For details, see [1].

Let us recall the following standard definitions. For $x \in X$, the isotropy subgroup of x is $T_x = \{t \in T \mid tx = x\}$. H^0 is the identity component of a subgroup H of T, and $F(H, X) = \{x \in X \mid hx = x$ for all $h \in H\}$, usually written just $F(H)$. For A an invariant subspace of X, A' is the corresponding subspace of the orbit space $X' = X/T$.

Let $X \in \mathscr{M}$ mean that X is a compact Z-cohomology manifold with $H^*(X; \mathbf{Q})$ finite dimensional. By Theorem 1.1, p. 85 of [1], this guarantees that any torus action on X will have a finite number of distinct isotropy subgroups. Let $X \sim Y$ mean $X \in \mathcal{M}$ and the rational cohomology rings of X and Y are the same. (The conditions on X are used only to ensure the existence of the Fary spectral sequence in Lemma 2.3, and any other conditions which do this would be sufficient.)

For a graded Q-module $A = \{A_i\}$, the Poincaré polynomial of A is defined to be $P(A) = \Sigma_i t^i \dim_{\mathbf{Q}}(A_i)$. For a topological space X, we set $P(X) = P(H^*(X))$. In particular, $P(X)(t = 1) = \dim_{\mathbf{Q}} H^*(X)$. The symbol lim will denote the limit of an expression as $t \to 1$, with $0 \leq t < 1$. For power series f and g, $f \preceq g$ means that each coefficient of f is less than or equal to the corresponding coefficient of g.

DEFINITION 2.1. *Let φ denote an action of a torus T of rank r on a space X. For k any integer, define*

$$f(\varphi, k) = \lim P(X_T)(1 - t^2)^{r-k}$$

if the limit exists. For a power series $f(t)$ in t, the order of the pole of $f(t)$ is defined to be the smallest integer n such that $\lim f(t)(1 - t^2)^n$ *exists.*

LEMMA 2.2. *For any action φ of T^r on $X \in \mathcal{M}$ and $S = \{x \in X \mid$*corank *$(T_x{}^0) < k\}$, $P((X - S)_T)$ has the form $p(t)(1 - t^2)^{-r}$, where $p(t)$ is a finite polynomial with integer coefficients.*

Proof. Since $H^*(X)$ is finite dimensional, the fixed point set of any torus acting on X has finite dimensional cohomology, so we can show, using Mayer–Vietoris sequences,that $H^*(S)$ is finite dimensional, so $H^*(X - S)$ is finite dimensional.

Now consider the Leray spectral sequence of $(X - S)_T \to B_T$. Each stage E_r is a graded module over the graded ring $H^*(B_T)$ in the usual way. The fact that $H^*(X - S)$ is finite dimensional means that E_2 is finitely generated over $H^*(B_T)$. Since $H^*(B_T)$ is Noetherian and $E_\infty = E_r$ for r large enough, E_∞ is also finitely generated over $H^*(B_T)$. (Everything taken in the graded sense, of course.)

Then by the Hilbert theorem on syzygies ([6], p. 217), E_∞ has a finite free resolution by finitely generated free $H^*(B_T)$-modules. Then $P((X - S)_T) = P(E_\infty)$ is the alternating sum of the Poincaré series of the modules in the resolution, which finishes the proof.

Note that this implies that if $P((X - S)_T)(1 - t^2)^{r-k}$ is bounded for $0 \leq t < 1$, then $f(\varphi \mid X - S, k)$ exists. This fact is used in the next lemma.

LEMMA 2.3. *Let φ denote an action of T^r on $X \in \mathcal{M}$ and $S = \{x \in X \mid corank\ (T_x{}^0) < k\}$. Then $P((X - S)_T)$ has pole of order $\leq r - k$, $f(\varphi \mid X - S, k)$ exists and $f(\varphi \mid X - S, k - 1) = 0$.*

Proof. Let $Y = X - S$. We look at the Fary spectral sequence of $Y_T \to Y'$. To describe this, let $U_1, \ldots, U_s$ be the identity components of all the isotropy subgroups occuring in the action of T on Y, ordered so that if rank $(U_i) >$ rank (U_j), then $i < j$. We have a filtration of Y

$$\varnothing = Y_0 \subseteq Y_1 \subseteq \cdots \subseteq Y_s = Y$$

where $Y_i - Y_{i-1} = \{y \in Y \mid (T_y)^0 = U_i\}$, and a corresponding filtration of Y'

$$\varnothing = Y_0{}' \subseteq Y_1{}' \subseteq \cdots \subseteq Y{}' = Y'$$

This gives rise to the Fary spectral sequence (E_r). We have

$$E_2^{p,q} = \Sigma_j H^p(Y'_j - Y'_{j-1}; H^q(B(U_j)))$$

and $(E_r) \Rightarrow H^*(Y_T)$. It follows that $0 \preceq P(Y_T) = P(E_\infty) \preceq P(E_2) = \Sigma_j P(Y'_j - Y'_{j-1})(1 - t^2)^{-(r-k_j)}$, where k_j is the corank of U_j, which is $\geq k$. Then $0 \leq P(Y_T)(1 - t^2)^{r-k} \leq \Sigma_j P(Y'_j - Y'_{j-1})(1 - t^2)^{k_j - k}$as functions of t for $t \in [0, 1)$. Thus $f(\varphi \mid Y, k)$ must exist, and in fact

$$0 \leq f(\varphi \mid X - S, k) \leq \Sigma_{k_j = k} P(Y'_j - Y'_{j-1})(t = 1).$$

THEOREM 2.4. *Suppose that φ is an action of a torus T on $X \in \mathcal{M}$. Let*

$$S = \{x \in X \mid \text{corank } (T_x{}^0) < k\}.$$

Let $U_1, \ldots, U_u$ be the identity components of isotropy subgroups of corank k. Then

$$f(\varphi \mid X - S, k) = \Sigma_{i=1}^u f(\varphi \mid F(U_i) - S, k).$$

Proof. Let $G = \{x \in X \mid \text{corank } (T_x{}^0) \leq k\}$. We have the exact sequence

$$H^n((X - S)_T) \xrightarrow{i^*} H^n((G - S)_T) \xrightarrow{\delta} H^{n+1}((X - G)_T) \xrightarrow{j^*} H^{n+1}((X - S)_T)$$

which gives

$$P((G - S)_T) - P((X - S)_T) = P(\text{coker } i^*) - P(\ker i^*).$$

We have induced maps $\bar{\delta}$ and j^*

$$\bar{\delta}\colon \text{coker } (i^*) \to H^*((X - G)_T) \quad \text{a monomorphism}$$

$$j^*\colon H^*((X - G)_T) \to \ker (i^*) \quad \text{an epimorphism.}$$

Therefore $tP(\text{coker } i^*)$ and $P(\ker i^*)$ are both $\preceq P((X - G)_T)$. But by 2.3, $f(\varphi \mid X - G, k)$ is zero. Therefore

$$\lim P(\text{coker } i^*)(1 - t^2)^{r-k} = \lim P(\ker i^*)(1 - t^2)^{r-k} = 0.$$

Therefore $f(\varphi \mid X - S, k) = f(\varphi \mid G - S, k)$. But $G - S$ is the disjoint union of the $F(U_i) - S$, so the conclusion follows.

We can interpret this theorem as saying more or less that the order of the pole of $P(X_T)$ and the coefficient of that pole are due entirely to the most singular points in X; that is, the points whose isotropy subgroups have maximum rank.

Theorem 2.4 and Lemma 2.3 enable us to find analogues of Borel's formula and to estimate $c(X)$, respectively. We use the Leray spectral sequence of $\pi\colon X_G \to B_G$ and simple algebraic arguments to calculate $f(\varphi, k)$.

COROLLARY 2.5. *Suppose that $X \in \mathcal{M}$. Suppose that there is an integer k_0 such that $P(X_T)$ has pole of order greater than or equal to $r - k_0$ for any action of a torus T of rank r. Then $c(X) \leq k_0$.*

To apply this, consider the spectral sequence $\{E_r\}$ of $\pi\colon X_T \to B_T$. We have $P(X_T) = P(E_\infty)$, while $P(E_2) = P(X)(1 - t^2)^{-r}$. Therefore *to show that $c(X) \leq k_0$, we need only show that in the process of going from E_2 to E_∞ we loose k_0 or less from the order of the pole.* This is done by brute force spectral sequence arguments. Our results here are contained in the following propositions.

PROPOSITION 2.6. *If $X \sim T^k$, then $c(X) \leq k$. If $X = T^k$, then $c(X) = k$.*

PROPOSITION 2.7. *If $X \sim S^p \times S^q$ with p and $q \geq 1$, then $c(X) \leq$ the number of p and q which are odd. If $X = S^p \times S^q$, then we have the corresponding equality for $c(X)$.*

Proof of 2.6. We choose bases so that in the Leray spectral sequence

$$E_2 = \Lambda(u_1, \ldots, u_k) \otimes \mathbf{Q}[x_1, \ldots, x_r]$$

and d_2 is given by

$$\begin{aligned} d_2 &: u_i \to x_i \qquad i = 1, \ldots, s \\ d_2 &: u_i \to 0 \qquad i = s+1, \ldots, k. \end{aligned}$$

Then

$$E_\infty \cong E_3 \cong \Lambda(u_{s+1}, \ldots, u_k) \otimes \mathbf{Q}[x_{s+1}, \ldots, x_r],$$

so $P(X_T) = P(E_\infty) = (1+t)^{k-s}(1-t^2)^{-(r-s)}$, which has pole of order $r - s$. But $s \leq k$, so $r - s \geq r - k$. Therefore $c(X) \leq k$.

Proposition 2.7 follows by easy spectral sequence arguments, using the following lemma which may be proved by induction on k. The proofs are left to the reader.

LEMMA 2.8. *If R is the graded ring $\mathbf{Q}[x_1, \ldots, x^r]$ with generators x_i of degree two and if $a_1, \ldots, a_k$ are any homogeneous elements of nonzero degree in R, then $P(R/(a_1, \ldots, a_k)) \geq p(t)(1-t^2)^{-(r-k)}$ as functions of t for $t \in [0, 1)$, where $p(t)$ is a finite nonzero polynomial with nonnegative coefficients.*

One might conjecture that if $H^*(X)$ is an exterior algebra on odd dimensional generators then $c(X)$ is less than or equal to the number of generators. This is disappointingly difficult to prove. Even if one assumes that the degrees of the generators are such that $H^*(X)$ is transgressively generated, one cannot in general get the necessary estimate of $P(E_\infty)$ by simple brute force methods.

If we apply Theorem 2.4 with $k = 0$ to an arbitrary torus action on $X \in \mathscr{M}$ we get the well known result $\dim H^*(X) \geq \dim H^*(F)$. If we apply Theorem 2.4 with $k = 1$ to an action on $X \sim S^n$, we get Borel's theorem: let $F = F(T) \sim S^s$, and $F(U_i) \sim S^{n_i}$. Then we can calculate from the Leray spectral sequence that $f(\varphi \mid X - F, 1) = (n-s)/2$, and $f(\varphi \mid F(U_i) - F, 1) = (n_i - s)/2$. (The method is the same used in the next section to prove Theorem 1.3, although the case here is much simpler.) Substitution into Theorem 2.4 gives $n - s = \Sigma(n_i - s)$, which is Borel's theorem.

It seems that this is the general procedure for applying Theorem 2.4. That is, suppose one has an action φ of a torus T on a space X of a certain cohomology type. Let $S = \{x \in X \mid \operatorname{corank}(T_x) < c(X)\}$. Then one has

$$f(\varphi \mid X - S, c(X)) = \Sigma f(\varphi \mid F(U_i) - S, c(X)) \tag{2.9}$$

where the U_i are the identity components of isotropy subgroups of corank $c(X)$. Finally one tries to express both sides of equation (2.9) in terms of the properties of X and of the fixed point sets of the subtori of coranks $0, \ldots, c(X)$ of T. This is what will be done in the next section for $X \sim S^p \times S^q$ with p and q odd.

The results will not always be as good as they are for $X \sim S^p$ or $X \sim S^p \times S^q$ with p and q odd. For example, suppose $X \sim S^p \times S^q$ with p odd and q even. Then $c(X) \leq 1$. Suppose T acts on X with $F(T) = \varnothing$. If $q > p$, our methods show only that $f(\varphi, 1)$ is either $p + 1$ or $q + 1$, depending on φ, and so there is a corresponding indefiniteness in the conclusion drawn from (2.9).

§3. PROOFS OF 1.2 AND 1.4

The following lemma is the essential part of the proof of 1.2.

LEMMA 3.1. *Suppose φ is an action of a torus T on $X \sim S^p \times S^q$ with p and q odd and $F(T) = \varnothing$. Let $U_1, \ldots, U_u$ be the corank one subtori of T that have fixed points. We have $F(U_i) \sim S^{p_i} \times S^{q_i}$ with p_i and q_i odd. Let $S = \cup_{i=1}^{u} F(U_i)$.* Then

$$f(\varphi \mid X - S, 2) = e(X) - \Sigma_{i=1}^{u} eF(U_i) + \Sigma_{i \neq j} r_i r_j$$

where $r_i = (p_i + 1)/2$, $i = 1, \ldots, u$.

Note. We have specified neither that $p_i \geq q_i$ nor that $p_i \leq q_i$. Therefore knowledge of $F(U_i)$ does not suffice to determine r_i. For example, if we know $F(U_1) \sim S^3 \times S^5$, we still cannot tell whether $r_1 = (3 + 1)/2$ or $(5 + 1)/2$.

The following lemma will be useful. The proof is left to the reader.

LEMMA 3.2. *For a, b, $\in R = \mathbf{Q}[x_1, \ldots, x_r]$, let $(a:b) = \{r \in R \mid br \in (a)\}$. Then for g a linear combination of $x_1, \ldots, x_r$ and x homogeneous in R,*

$$(g^s : x) = (g^{s-t})$$

where t is found as follows: express x in terms of any set of generators of R which includes g. Then t is the lowest power to which g occurs in any term of x.

Proof of 3.1. We will divide the proof into two cases: first the case $S = \varnothing$ and second the more complicated case $S \neq \varnothing$. The core of the method is the fact that knowledge of the order of the pole of E_∞ gives one a surprising amount of control over the nature of E_∞. R will denote the ring $H^*(B_T)$, and $|a|$ the degree of a homogeneous element of R. The rank of T is r.

First take the case $S = \varnothing$. We will show that $f(\varphi, 2) = e(X)$ by showing that $P(X_T) = (1 - t^{q+1})(1 - t^{p+1})(1 - t^2)^{-r}$.

First take $p < q$ and look at the spectral sequence (E_r) of the filtration $X_T \to B_T$. By the evenness of $H^*(B_T)$, the generators u and v of $H^p(X)$ and $H^q(X)$ are transgressive. Further, neither of the transgressions a and b of u and v can be zero. (If both a and b were zero, then the spectral sequence would collapse, $P(E_\infty)$ would have pole of order r, and $F(T)$ would be nonempty by 2.3. If one of a or b were zero, then it is easy to see that $P(E_\infty)$ would have pole of order $r - 1$, so that by 2.3 S would be nonempty.)

The E_∞ term of the spectral sequence must be zero except in rows zero and q. Row zero is $R/(a, b)$, and row q is $\cong v \otimes ((a:b)/(a))$. Then

$$P(X_T) = P(E_\infty) \geqq P(\text{row } q \text{ of } E_\infty) = t^q[P((a:b)) - P((a))].$$

Let d be the gcd of a and b in R and write $a = cd$ and $b = de$. Then $(a : b) = (c)$, so

$$P(X_T) \geqq t^q(t^{|c|} - t^{|a|})(1 - t^2)^{-r}.$$

It is easy to see that this has pole of order $r - 1$, unless it is zero. But by 2.3 and the hypothesis $S = \varnothing$, $P(X_T)$ must have pole of order $\leq r - 2$. Therefore we must have $|c| = |a|$; that is, a and b must have no common factors in R, and row q of E_∞ is zero.

Therefore
$$\begin{aligned} P((a, b)) &= P((a)) + P((b)) - P((a) \cap (b)) = P((a)) + P((b)) - P((ab)) \\ &= [t^{|a|} + t^{|b|} - t^{|a|+|b|}](1 - t^2)^{-r}, \end{aligned}$$

and

$$P(R/(a, b)) = P(R) - P((a, b)) = (1 - t^{|a|})(1 - t^{|b|})(1 - t^2)^{-r}.$$

But $|a| = p + 1$, $|b| = q + 1$, so

$$P(X_T) = P(E_\infty) = P(R/(a, b)) = (1 - t^{p+1})(1 - t^{q+1})(1 - t^2)^{-r}.$$

When $p = q$ or $p > q$, the argument is entirely similar and will be left to the reader. This finishes the case $S = \varnothing$.

Now we treat the case $S \neq \varnothing$. We have the cohomology exact sequence

$$H^*(X_T) \xrightarrow{j^*} H^*(S_T) \xrightarrow{\delta} H^*((X - S)_T) \to H^*(X_T) \xrightarrow{j^*} H^*(S_T)$$

which yields

$$P((X - S)_T) = P(\ker j^*) + tP(\operatorname{coker} j^*).$$

We will find $P((X - S)_T)$ essentially by finding j^*. Note that by 2.3, $P((X - S)_T)$ must have pole of order $\leq r - 2$, and so in turn $\ker(j^*)$ and $\operatorname{coker}(j^*)$ must have poles of order $\leq r - 2$.

First look at the spectral sequence of $X_T \to B_T$. The generators u and v of $H^p(X)$ and $H^q(X)$ transgress to a and b. As before, let d be the gcd of a and b and write $a = cd$, $b = de$. For $a \neq 0$ and $p < q$, the E_∞ term is shown in Fig. 1.

$$\begin{array}{c|l} q & \text{---}\ v \otimes (c)/(a) \ \text{————} \\ & \\ 0 & \text{---}\ R/(a, b) \ \text{————} \end{array}$$

FIG. 1. $E_\infty(X_T)$

For other cases ($p < q$ and $a = 0$, or $p = q$, or $p > q$) the E_∞ term can again be put in this form, possibly by interchanging p and q, u and v, etc. (For the case $p = q$, the row p of $E_\infty(X_T)$ is $(eu - cv)/(bu - av)$. It will be seen that this is sufficiently like the form of Fig. 1 for our purposes.)

Now let us look at S_T. Let $i = 1, \ldots, u$. Let $F_i = F(U_i) \sim S^{p_i} \times S^{q_i}$. S is the disjoint union of the compact spaces F_i. We wish to look at the spectral sequence of $(F_i)_T \to B_T$. Because U_i acts trivially on F_i, we have an action of T/U_i on F_i. Note that $T/U_i \cong S^1$. The homomorphism $T \to T/U_i$ commutes with the actions on F_i, so we have a commuting diagram of fibrations

$$\begin{tikzcd} (F_i)_T \arrow[r] \arrow[d] & (F_i)_{T/U_i} \arrow[d] \\ B(T) \arrow[r, "f_i"] & B(T/U_i). \end{tikzcd}$$

The (ring) generators of $H^*(F_i)$ are transgressive in both spectral sequences, and transgression commutes with induced maps. Let h_i be a generator of $H^2(B(T/U_i))$; let $g_i = f_i^*(h_i)$. Then the transgressions of the generators of $H^*(F_i)$ in the spectral sequence of $(F_i)_T \to B_T$ must be in the subring of $H^*(B(T))$ generated by g_i.

Note that if we interpret g_i as an element of $H^1(T)$, which in turn is isomorphic to a subgroup of the dual space of the Lie algebra of T, we find that g_i is a linear functional which defines the subtorus U_i of T.

Now let us take $p_i < q_i$ for the moment. If u_i, the generator of $H^{p_i}(F_i)$, has nonzero transgression in the spectral sequence of $(F_i)_T \to B_T$, then, up to a scalar factor, it transgresses to $g_i^{r_i}$. Since v_i, the generator of $H^{q_i}(F_i)$, must transgress into the subring of $H^*(B(T))$ generated by g_i, it is easy to see that E_∞ must be as shown in Fig. 2.

$$\begin{array}{r|l} q_i & \text{——}\ v_i \otimes R/(g_i^{r_i})\ \text{——} \\ & \\ 0 & \text{——}\ R/(g_i^{r_i})\ \text{——} \end{array}$$

FIG. 2. $E_\infty((F_i)_T)$

On the other hand, if u_i has zero transgression, then v_i must have nonzero transgression, namely $g_i^{s_i}$, where $s_i = (q_i + 1)/2$, up to a scalar multiple. (If both transgressions were zero, $F(T)$ would be nonempty by 2.3.) Then $E_\infty((F_i)_T)$ is nonzero only in rows zero and p_i; row zero is $R/(g_i^{s_i},)$ and row p_i is $u_i \otimes R/(g_i^{s_i})$. By interchanging p_i and q_i, r_i and s_i, u_i and v_i, we can consider this to be the same as Fig. 2. (This interchange is what makes it impossible to identify r_i from knowledge of the F_i, as was explained immediately after the statement of this lemma.) Similarly for $p_i = q_i$, we can consider $E_\infty((F_i)_T)$ to be given by Fig. 2.

We can now consider $j^* : H^*(X_T) \to H^*(S_T) = \oplus H^*((F_i)_T)$ to be a map from Fig. 1 to the direct sum of u copies, $i = 1, \ldots, u$, of Fig. 2.

Note that in each E_∞ there is only one nonzero entry in each total degree, since q and the q_i are all odd. Therefore we don't have to worry about the relation between $H^*(X_T)$ and $H^*(S_T)$ and their associated gradeds.

For convenience, let

$$\Phi = j^* \mid H^{\text{odd}}(X_T) = j^* \mid \text{top row of } E_\infty(X_T)$$
$$\Psi = j^* \mid H^{\text{even}}(X_T) = j^* \mid \text{bottom row of } E_\infty(X_T).$$

We have already mentioned that because $P((X - S)_T)$ has pole of order $\leq r - 2$, $P(\ker j^*)$ and $P(\operatorname{coker} j^*)$ must have poles of order $\leq r - 2$. It follows that the Poincaré series of ker and coker of Φ and Ψ must have poles of order $\leq r - 2$.

Consider first Ψ. Ψ is the map

$$R/(a, b) \to R/(g_1^{r_1}) \oplus \cdots \oplus R/(g_u^{r_u})$$

given by

$$[1] \to [1] \oplus \cdots \oplus [1],$$

by the commutativity of

$$\begin{tikzcd} H^*(X_T) \arrow[r, "j^*"] & H^*(S_T) \\ H^*(B_T) \arrow[u, "\pi^*"] \arrow[r, equal] & H^*(B_T) \arrow[u, "\pi^*"] \end{tikzcd}.$$

Consider now

$$\begin{tikzcd} R \arrow[r, "\Psi_0"] \arrow[d] & R/(g_1^{r_1}) \oplus \cdots \oplus R/(g_u^{r_u}) \\ R/(a, b) \arrow[ur, "\Psi"'] & \end{tikzcd}$$

where the vertical map is the quotient map and Ψ_0 is the obvious map making the diagram commute. Then

$$\ker(\Psi_0) = (g_1^{r_1}) \cap \cdots \cap (g_u^{r_u}) = (g_1^{r_1} \cdots g_u^{r_u}),$$

since the g_i's are distinct primes. Then $P(\operatorname{im} \Psi_0) = (1 - t^{2(r_1 + \cdots + r_u)})(1 - t^2)^{-r}$, so

$$P(\operatorname{coker} \Psi) = P(\operatorname{coker} \Psi_0) = [(1 - t^{2r_1}) + \cdots + (1 - t^{2r_u}) - (1 - t^{2(r_1 + \cdots + r_u)})](1 - t^2)^{-r}$$

and we find easily that

(3.3) $$\lim P(\operatorname{coker} \Psi)(1 - t^2)^{r-2} = \Sigma_{i<j} r_i r_j.$$

Also, $\ker \Psi = (\ker \Psi_0)/(a, b)$, so

$$P(\ker \Psi) = P(\ker \Psi_0) - P((a, b)) = [t^{2(r_1 + \cdots + r_u)} - (t^{|a|} + t^{|b|} - t^{|a|+|b|-|d|})](1 - t^2)^{-r},$$

where we recall that d is the gcd of a and b. It is easy to see that because $P(\ker \Psi)$ must have pole of order $\leq r - 2$, we must have $2(r_1 + \cdots + r_u) = |d|$. And, taking this value of $|d|$, we find

(3.4) $$\lim P(\ker \Psi)(1 - t^2)^{r-2} = (r_1 + \cdots + r_u)^2 + r_0 s_0 - (r_0 + s_0)(r_1 + \cdots + r_u),$$

where $r_0 = (p + 1)/2$, $s_0 = (q + 1)/2$, $r_0 s_0 = e(X)$.

Now look at Φ. We will show that Φ is in fact a monomorphism. Let Φ_i be Φ followed by the projection of $H^*(S_T)$ onto $H^*((F_i)_T)$. Then Φ_i may be regarded as a map

$$\Phi_i\colon (c)/(a) \to R/(g_i^{r_i})$$

which increases degree by $q - q_i$. Let $\Phi_i(c) = x_i + (g_i^{r_i})$. Then clearly

$$\ker(\Phi_i) = [c(g_i^{r_i} : x_i)]/(a).$$

By 3.2, $(g_i^{r_i} : x_i) = (g_i^{w_i})$, where $0 \le w_i \le r_i$. Then

$$\begin{aligned}\ker(\Phi) &= \ker(\Phi_1) \cap \cdots \cap \ker(\Phi_u)\\ &= \{c \cdot [(g_1^{w_1}) \cap \cdots \cap (g_u^{w_u})]\}/(a)\\ &= (cg_1^{w_1} \cdot \cdots \cdot g_u^{w_u})/(a),\end{aligned}$$

since the g_i's are distinct primes. Thus

$$P(\ker \Phi) = [t^{|c| + 2(w_1 + \cdots + w_u)} - t^{|a|}](1 - t^2)^{-r}.$$

This has pole of order $r - 1$ unless $|a| = |c| + 2(w_1 + \cdots + w_u)$ and $P(\ker \Phi)$ is zero. Since $P(\ker \Phi)$ must have pole of order $\le r - 2$, we must have $|a| = |c| + 2(w_1 + \cdots + w_u)$ and Φ must be a monomorphism. Since we have $|a| = |c| + |d|$, we get $|d| = 2(w_1 + \cdots + w_u)$. But we found previously that $|d| = 2(r_1 + \cdots + r_u)$, so we must have $w_i = r_i$, $i = 1, \ldots, u$. The fact that $\Phi_i(c) = x_i + (g_i^{r_i})$ in $R/(g_i^{r_i})$ defines a map from $(c)/(a)$ into $R/(g_i^{r_i})$ implies that $g_i^{r_i} \mid dx_i$. Since $(g_i^{r_i} : x_i) = (g_i^{r_i})$, $g_i \nmid x_i$, so $g_i^{r_i} \mid d$. Since $|d| = 2(r_1 + \cdots + r_u)$, we know that up to a rational factor,

$$d = g_1^{r_1} \cdot \cdots \cdot g_u^{r_u}. \tag{3.5}$$

Note. We should point out that this last equation is of interest in itself: it says that (in the case $F(T) = \emptyset$) the corank one subtori that have fixed points, as well as a partial description of the fixed point sets, are given by the prime factorization of the gcd of the transgressions of the generators of $H^*(X_T)$.

Using the fact that Φ is a monomorphism and our expression for $|d|$, it is easy to calculate

$$\begin{aligned}P(\operatorname{coker} \Phi) &= P(\text{codomain } \Phi) - P(\text{domain } \Phi)\\ &= \Sigma_i P(\text{top row of } E_\infty((F_i)_T)) - P(\text{top row of } E_\infty(X_T))\\ &= \{\Sigma_i[t^{q_i}(1 - t^{2r_i})] - t^q(t^{|c|} - t^{|a|})\}(1 - t^2)^{-r}\\ &= \{\Sigma_i[t^{q_i}(1 - t^{2r_i})] - t^{q + |a| - |d|}(1 - t^{|d|})\}(1 - t^2)^{-r}\end{aligned}$$

and we find $\lim P(\operatorname{coker} \Phi)(1 - t^2)^{r-2} =$

$$(r_0 + s_0)(r_1 + \cdots + r_u) - \tfrac{1}{2}\Sigma r_i^2 - \Sigma r_i s_i - \tfrac{1}{2}(r_1 + \cdots + r_u)^2 \tag{3.6}$$

where $s_i = (q_i + 1)/2$. Adding together (3.3), (3.4) and (3.6), we get the conclusion of Lemma 3.1.

Note. Suppose we pick some i and look at $F_i = F(U_i) \sim S^m \times S^n$, with m and n odd. Suppose we want to find out which of the two possibilities $(m + 1)/2$ or $(n + 1)/2$ is the number r_i which appears in the statement of Lemma 3.1. By the discussion associated with Fig. 2, this depends only on which of the two ring generators of $H^*(F_i)$ transgresses to zero in the spectral sequence of $(F_i)_{S^1} \to B_1$, where S^1 has the action from $S^1 = T/U_i$.

Proof of 1.2. We have $U_1, \ldots, U_u$ the identity components of the corank one isotropy subgroups and $U_{u+1}, \ldots, U_w$ the identity components of the corank two isotropy subgroups.

Denote these $H_1, \ldots, H_u$ and $K_1, \ldots, K_v$ respectively. We have $F(H_i) = S^{p_i} \times S^{q_i}$ and $r_i = (p_i + 1)/2$. $S = \cup_{i=1} F(H_i)$. By 2.4, we have

(3.7) $$f(\varphi \mid X - S, 2) = \Sigma_{n=1}^{v} f(\varphi \mid F(K_n) - S, 2).$$

By 3.1,

$$f(\varphi \mid X - S, 2) = e(X) - \Sigma_{i=1}^{u} eF(H_i) + \Sigma_{i \neq j} r_i r_j .$$

We can also apply 3.1 to each term on the right side of 3.7, since each $F(K_n)$ is itself (cohomologically) a product of two odd spheres. Now, the identity components of corank one isotropy subgroups of the restricted action $\varphi \mid F(K_n)$ are just those H_i's which include K_n. (To see this, suppose $x \in F(K_n) \cap F(H)$, where H has corank one. Then $x \in F(H \cdot K_n)$. If $K_n \not\subset H$, then $H \cdot K_n = T$, which contradicts the hypothesis $F(T) = \varnothing$.) Note that if $H_i \supseteq K_n$, then $F(H_i, F(K_n, X)) = F(H_i, X)$. Thus

$$f(\varphi \mid F(K_n) - S, 2) = eF(K_n) - \Sigma_{H_i \supseteq K_n} eF(H_i) + \Sigma\{r_i r_j \mid i \neq j;\ H_i, H_j \supseteq K_n\}.$$

Putting this in (3.7), we see that we need to show that

$$\Sigma_{\text{all } i \neq j} r_i r_j = \Sigma_{n=1}^{v} [\Sigma\{r_i r_j \mid i \neq j;\ H_i, H_j \supseteq K_n\}].$$

But, first of all, by the note after the proof of 3.1, the symbol r_i stands for the same number on both sides. Secondly, we claim that for any pair $i \neq j$, there is exactly one n such that H_i and $H_j \supseteq K_n$.

To see this, let $K = (H_i \cap H_j)^0$. This is of corank two; we have only to show that it is one of the K_n's; that is, that it is the identity component of some isotropy subgroup. We know that $F(K) \not\subseteq F(H_i)$, since $F(K) \supseteq F(H_j)$. For $k \neq i$, $F(K) \not\subseteq F(H_k)$, since if $F(H_k) \supseteq F(K) \supseteq F(H_i) \neq \varnothing$, then we have $F(H_k \cdot H_i) = F(H_i) \neq \varnothing$ and $H_k \cdot H_i = T$, contrary to the hypothesis $F(T) = \varnothing$. Since there are a finite number of H_k, since $F(K) \not\subseteq$ any $F(H_k)$, and since we are dealing with connected cohomology manifolds, it follows that $F(K)$ is not contained in $S = \cup F(H_k)$. Take $x \in F(K) - S$. Clearly $K = (T_x)^0$.

We have almost finished the proof of 1.2. We have proven the formula of 1.2, with the sums taken over the identity components of isotropy subgroups of coranks one or two. We wish to show that this implies the theorem as stated, with the sums taken over general subtori of coranks one or two. This fact, as well as our comment (immediately before the statement of 1.4) that the summations in 1.2 and 1.3 need only be taken over *distinguished* subtori of coranks one and two, will follow from the following two lemmas. Since we intend these to apply to 1.3 as well as 1.2, we do not assume $F(T) = \varnothing$.

LEMMA 3.8. *Suppose T acts on $X \sim S^p \times S^q$, p and q odd. If H is a corank one subtorus of T with $eF(H) - eF(T) \neq 0$, then H is distinguished. If K is a corank two subtorus of T with*

(3.9) $$eF(K) - eF(T) - \Sigma_{H \supseteq K}[eF(H) - eF(T)]$$

nonzero, then K is distinguished.

LEMMA 3.10. *Suppose T acts on $X \sim S^p \times S^q$, p and q odd. Then a subtorus of T is distinguished if and only if it is the identity component of some isotropy subgroup.*

Proof of 3.8. The statement for corank one subtori will be left to the reader. Suppose that K is a corank two subtorus such that (3.9) is nonzero. We must certainly have $F(K) \neq F(T)$. Suppose that there is a corank one subtorus $H \supseteq K$ in T with $F(H) = F(K)$. If H' were another corank one subtorus with $F(T) \subsetneqq F(H') \subseteq F(K) = F(H)$, we would have $F(T) = F(H \cdot H') \supseteq F(H') \supsetneqq F(T)$, which is impossible; thus H is the only corank one subtorus which includes K and has $F(H) \neq F(T)$. This shows that (3.9) is zero, a contradiction.Therefore $F(H) \subsetneqq F(K)$ for each corank one H with $H \supseteq K$, so K is distinguished.

Proof of 3.10. It is easy to see that the identity component of an isotropy subgroup must be distinguished. On the other hand, suppose K is a distinguished subtorus. Let $L_1, \ldots, L_k$ be the identity components of isotropy subgroups of points in $F(K)$. We have $K \subseteq L_i$ for all i. We claim $K = L_i$ for some i. Suppose not. For all i, $K \subsetneqq L_i$, so, since K is distinguished, $F(K) \supsetneqq F(L_i)$. Since we are dealing with connected (cohomology) manifolds, we get $F(K) \supsetneqq \cup F(L_i)$. Take $x \in F(K) - \cup F(L_i)$. Clearly $K = (T_x)^0$.

Proof of 1.4. Let C be the ineffective kernel. Let H denote a general distinguished subtorus of corank one, let K denote a general distinguished subtorus of corank two, and let L denote a general distinguished subtorus of corank either one or two. Let $M = (\cap L)^0$. We want to show $M = C^0$.

It is easy to see that $M \supseteq C^0$, since if $L \not\supseteq C^0$, then $L \cdot C^0$ is a subtorus which strictly includes L, but $F(L \cdot C^0) = F(L)$, contrary to the fact that L is distinguished.

To complete the first part of the theorem, we want to show $M \subseteq C$; that is, M acts trivially on X. First consider the case $F(T) = \varnothing$. Let $Y = F(M)$. We want to show that $Y = X$. We must have $Y \sim S^i \times S^j$, i and j odd, and by equation (1.6) of [3], we may take $i \leq p$ and $j \leq q$. Then $e(Y) \leq e(X)$, and if $e(Y) = e(X)$, we must have $\dim(Y) = \dim(X)$, so $Y = X$. (This is where we need X to be orientable.)

Since M is contained in any L, we must have that the distinguished subtori L of corank one and two of the action of T on X and on Y are the same, and $F(L, X) = F(L, Y)$. Applying 1.2 to the actions on X and Y we get

$$e(X) - \Sigma_H eF(H) = \Sigma_K[eF(K) - \Sigma_{H \supseteq K} eF(H)]$$
$$e(Y) - \Sigma_H eF(H) = \Sigma_K[eF(K) - \Sigma_{H \supseteq K} eF(H)].$$

Thus $e(X) = e(Y)$ and $X = Y$.

In case $F(T) \neq \varnothing$, we cannot use 1.2. However, the arguments described in section four show that

$$\dim(X) - \dim F(T) = \Sigma_H[\dim F(H) - \dim F(T)].$$

(This equality is also given by a theorem of Borel ([1], p. 182), if we further assume that X is first-countable. The argument described in section four gives this equality without further assumptions.) An argument like the one in the case $F(T) = \varnothing$ shows that $(\cap H)^0$ acts trivially, so certainly $M = (\cap L)^0$ acts trivially. This finishes the proof of the first part of 1.4.

For the second part of 1.4, take $x \in X$, and let $Y = F(T_x{}^0)$. Let $\varphi \mid Y$ be the action of T on Y. The identity component of the ineffective kernel of $\varphi \mid Y$ is easily seen to be $T_x{}^0$. Therefore, by the first part of the theorem, T_x^0 is the identity component of the intersection of the distinguished subtori of coranks one and two of $\varphi \mid Y$. But it is easy to see that these are the distinguished subtori of corank one and two of φ which contain T_x^0. This finishes the proof of 1.4.

§4. DESCRIPTION OF WORK ON CONJECTURE 1.3

As before, we have a torus T action φ on $X \sim S^p \times S^q$, p and q odd. Take $F = F(T, X)$ and $S = \{x \in X \mid \text{corank}\,(T_x^0) \leq 1\}$. To prove conjecture 1.3, it will be enough to determine $f(\varphi \mid X - S, 2)$. We expect that $f(\varphi \mid X - S, 2)$ will equal $e(X) - e(F) - \Sigma[e(F(U)) - e(F)]$ plus certain "junk" terms which correspond to the term $\Sigma_{i \neq j} r_i r_j$ in Lemma 3.1. (Here U runs over the identity components of the corank one isotropy subgroups.)

We calculate $P((X - S)_T)$ by calculating j^* in the exact sequence

$$H^*((X-F)_T) \xrightarrow{j^*} H^*((S-F)_T) \to H^*((X-S)_T) \to H^*((X-F)_T) \xrightarrow{j^*} H^*((S-F)_T).$$

We calculate $H^*((X - F)_T)$ from the exact sequence

$$H^*(X_T) \xrightarrow{j_0^*} H^*(F_T) \to H^*((X-F)_T) \to H^*(X_T) \xrightarrow{j_0^*} H^*(F_T),$$

and, since $S - F$ is the disjoint union of the $F(U) - F$, we can calculate $H^*((S - F)_T)$ from the exact sequences

$$H^*(F(U)_T) \xrightarrow{j_U^*} H^*(F_T) \to H^*((F(U)-F)_T) \to H^*(F(U)_T) \xrightarrow{j_U^*} H^*(F_T).$$

At each stage, we use the various bounds that 2.3 puts on the orders of the poles of the various Poincaré series. It turns out that the maps j_0^* and j_U^* are monomorphisms, so that we can write down the map $j^*\colon H^*((X - F)_T) \to H^*((S - F)_T)$ in quite an explicit form. However, we have been able to calculate the Poincaré series of $\ker(j^*)$ and $\text{coker}(j^*)$ only in the case in which the subtori U are independent in the sense that the corresponding elements $g_U \in H^1(T) \cong H^2(B_T)$ are independent. Thus conjecture 1.3 has been proven in this very special case, but the general case seems to require a great deal of analysis of the behavior of the Poincaré series of graded modules. The author will be happy to send any interested reader a more detailed writeup of the work described above.

In conclusion, let us note an important point in which the approach taken here differs from that of the original proof of Borel's Theorem. In Borel's proof, one first uses the exact sequence of the pair (X, F) to find $H^*(X - F)$, and then calculates $H^*((X - F)_T)$ using the spectral sequence of $(X - F)_T \to B_T$. This succeeds in Borel's case, because both X and F are spheres, so there is only one possibility for $H^*(X - F)$. However, in more complicated cases, one cannot calculate $H^*(X - F)$ merely from the exact sequence of the pair (X, F). Roughly what we have done instead is first calculated $H^*(X_T)$ and $H^*(F_T)$ from the spectral sequences of $X_T \to B_T$ and $F_T \to B_T$, and then used the exact sequence of the pair (X_T, F_T) to calculate $H^*(X_T - F_T) = H^*((X - F)_T)$. The reason for the relative success of this method is very roughly that there may be many ways to embed F in X purely as topological spaces, but there are many fewer ways to make F the fixed point set of a torus action on X. And, of course, the same things hold for $S = \{x \in X \mid \text{corank}(T_x^0) < k\}$ in place of F.

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